Itay Neeman. *Forcing with sequences of models of two types.* **Notre Dame J. Formal Logic**, vol. 55 (2014), pp. 265-298.

Neeman presents a new notion of forcing with finite sequences of models as side conditions, $\mathbb{P}_{\text{side}}$. The side conditions employ models of two types: countable and transitive. This forcing has numerous important applications including much simpler proofs of some classical results and new ways of collapsing cardinals, adding square sequences, and specializing trees. But most strikingly, with this new forcing Neeman shows a new proof of the consistency of the Proper Forcing Axiom (PFA), one where finite support iteration is used instead of countable support. This novel approach is key to obtaining higher analogues of PFA, which has been a long standing problem. More generally, there are many known results about objects of size ω_1, that until recently have remained intractable when generalized to ω_2, but finally can be addressed using this new type of forcing. This makes Neeman's work one of the most important recent developments in forcing.

There is a long history of using models as side conditions. The first use goes back to Todorcevic in *Partition problems in topology*, volume 84 of **Contemporary Mathematics**, AMS (1989). Then Mitchell in *Adding closed unbounded subsets of ω_2 with finite forcing*, **Notre Dame J. of Formal Logic**, 46(3):357-371 (2005) and Friedman in *Forcing with finite conditions*, **Set theory**, Trends Math., pgs 285-295 (2006) independently came up with a forcing with finite conditions that adds a club of order type ω_2. All of these approaches employ countable models as side conditions in order to ensure properness needed for preservation of cardinals. Taking this to a whole new level, Neeman's forcing uses models of two types: countable and transitive models. The author obtains the same applications as Mitchell and Friedman, but his approach significantly simplifies the original posets. Another important application is a way to collapse cardinals without adding certain branches through trees from the ground model. And most importantly, the author gives a new proof of the consistency of PFA, clearing the way to establishing the consistency of forcing axioms for posets with a higher analogue of properness, asserting meeting more than ω_1 many dense sets.

Let $\mathcal{K}$ be a transitive set that satisfies a large enough fragment of set theory, for example, $ZFC - P$. Let $\mathcal{S}$ be a set of countable elementary substructures of $\mathcal{K}$ and $\mathcal{T}$ be a set of transitive elementary substructures of $\mathcal{K}$. There are additional requirements on $\mathcal{S}$ and $\mathcal{T}$ that are listed in Definition 2.2 of the paper. The members of $\mathcal{S}$ and $\mathcal{T}$ are referred to as nodes. A condition of $\mathbb{P}_{\text{side}}$ is a finite set of nodes that forms an ϵ-increasing chain and is closed under intersections in the following way: if M, W are in s, where M is countable and W is transitive, then $M \cap W \in s$. The order of $\mathbb{P}_{\text{side}}$ is reverse inclusion. So a forcing condition can be extended by adding nodes above, below or between existing nodes. In applications, $\mathcal{S}$ and $\mathcal{T}$ will be stationary; moreover the forcing can be generalized by replacing "countable" in the nodes of $\mathcal{S}$ with "small", i.e. of size less than some fixed

κ. This is the case in the application of collapsing cardinals without adding branches.

The key property of the poset is the so called *residue lemma*, which ensures that the forcing is strongly proper. For a condition s and $Q \in s$, define $res_Q(s) := s \cap Q$. Then $res_Q(s) \in \mathbb{P}_{\text{side}}$, and the residue lemma states that if $t \in Q \cap \mathbb{P}_{\text{side}}$ and $t \leq res_Q(s)$, then s and t are compatible. Moreover, if Q is a transitive node, then $s \cup t$ is the common extension; and if Q is a countable node, then the common extension is the closure under intersections of $s \cup t$. Note that applying the residue lemma to $s = \{Q\}$ yields that for every $t \in Q$, there is $r \leq t$ with $Q \in r$. Another consequence is that if s is a condition and $Q \in s$, then s is a *strong master condition* for Q, i.e. it forces that $\dot{G} \cap Q$ is generic over V for $\mathbb{P}_{\text{side}} \cap Q$. It follows that the forcing is strongly proper for $\mathcal{S} \cup \mathcal{T}$, meaning that for every $Q \in \mathcal{S} \cup \mathcal{T}$, for every condition $t \in Q$, there is $r \leq t$, such that r is a strong master condition for Q. Then assuming stationarity of $\mathcal{S}$ and $\mathcal{T}$ gives preservation of desired cardinals.

The author shows $\mathbb{P}_{\text{side}}$ can be used to obtain some classical results of Mitchell and Friedman in a much simpler way. The first application is getting the tree property at $\aleph_2$, originally due to Mitchell: start with a weakly compact κ and use as nodes elementary substructures of $H(\kappa)$ for $\mathbb{P}_{\text{side}}$. More precisely, $\mathcal{S}$ consists of countable elementary submodels of $H(\kappa)$, and $\mathcal{T}$ consists of transitive elementary submodels W of $H(\kappa)$ such that $|W| < \kappa$ and W is countably closed. The strong properness of $\mathbb{P}_{\text{side}}$ for both types of nodes ensures that ω_1 and κ are preserved. Also if G is generic for $\mathbb{P}_{\text{side}}$, let A be the set of all transitive nodes in $\bigcup G$. Then A is increasing with respect to $\in$ and $\subset$, and is unbounded in $H(\kappa)$. For every $W \in A$, let W^+ denote its immediate successor in A. Then let B_W be the set of countable nodes in A between W and W^+. This is an $\in$-increasing sequence of countable elements, whose union is W^+, thereby witnessing that $|W^+|$ is collapsed to ω_1. Then κ becomes ω_2. The proof of the tree property uses the weak compactness of κ, and the fact the the forcing can be factored along a suitable transitive node, such that the factor poset is also strongly proper, and does not add new branches. Another application of $\mathbb{P}_{\text{side}}$ is a very nice way of adding a club with finite conditions through a regular $\theta \geq \omega_2$, originally independently obtained by Mitchell and Friedman with finite conditions using only countable models.

A new important application is collapsing cardinals without adding branches through trees. More precisely, let $\kappa^+ < \lambda = |H(\lambda)|$ be regular cardinals, such that $\kappa^{<\kappa} = \kappa$ and, if $\delta < \lambda$, then $\delta^\kappa < \lambda$. The small nodes are of the form $M \prec H(\lambda)$, $|M| \leq \kappa$, $\kappa \in M$ and $M^{<\kappa} \subset M$. The transitive nodes are of the form $W \prec H(\lambda)$, $|W| < \lambda$, $W^\kappa \subset W$. In this application, the author takes $\mathbb{P}_{\text{side}}$ to consist of conditions of length strictly less than kappa (rather than finite length as above). This forcing is κ-closed and, by strong properness, κ^+ and λ are preserved, and so it has the same effect on cardinal preservations as $Col(\kappa^+, < \lambda)$. But unlike the Levy collapse, it adds no branches of length κ^+ to trees from the ground model. The proof of that uses the fact that the forcing is strongly proper for the set of small nodes and the stationarity of the latter. This way of collapsing is extremely useful when dealing with arguments involving the tree property.

In a subsequent paper, *Two applications of finite side conditions at* ω_2, Neeman shows how to add a $\square_{\omega_1}$-sequence, and other variants of square, by a strongly proper forcing with finite side conditions. A similar result was independently obtained by Krueger. The author also shows how to add a weak specializing function for trees of height ω_2. Both results fit nicely into the project of using proper forcing to develop the theory of structures of size ω_2.

Another important application of Neeman's forcing method is a new proof that PFA is consistent relative to the existence of a supercompact cardinal. The standard proof uses the fact that countable support iterations preserve properness. Neeman's proof uses finite support iterations and relies on side conditions to ensure properness. Others had attempted without success to generalize properness to uncountable models and prove the consistency of a corresponding forcing axiom for more than ω_1 many dense sets. The standard consistency proof for PFA does not adapt but Neeman's does.

The set up of Neeman's proof is as follows. Suppose κ is a supercompact cardinal and $f : \kappa \to H(\kappa)$ is a Laver function. The transitive nodes are countably closed nodes of the form $H(\alpha)$, such that $(H(\alpha); f \restriction \alpha) \prec (H(\kappa); f)$, and the countable nodes are just elementary substructures of $(H(\kappa); f)$. Let $\mathbb{P}_{\text{side}}$ be the forcing with side conditions using these two types of nodes. The main forcing $\mathbb{A}$ consists of conditions $\langle s, p\rangle$, where $s \in \mathbb{P}_{\text{side}}$ and p is function whose domain is a finite subset of κ such that for every $\alpha \in \text{dom}(p)$, the following holds:

(1) $\Vdash_{\mathbb{A}\cap H(\alpha)} F(\alpha)$ is a proper poset and $p(\alpha) \in F(\alpha)$;
(2) $H(\alpha) \in s$;
(3) for countable $M \in s$ with $\alpha \in M$,

$$\langle s \cap H(\alpha), p \restriction \alpha\rangle \Vdash_{\mathbb{A}\cap H(\alpha)} p(\alpha) \text{ is a master condition for } M[\dot{G}_\alpha].$$

In item (3), $\dot{G}_\alpha$ is the name for the $\mathbb{A} \cap H(\alpha)$- generic filter. The order on conditions is $\langle s', p'\rangle \leq \langle s, p\rangle$ iff $s' \leq s$ and for all $\alpha \in \text{dom}(p) \subset \text{dom}(p')$, $\langle s' \cap H(\alpha), p' \restriction \alpha\rangle \Vdash_{\mathbb{A}\cap H(\alpha)} p'(\alpha) \leq_{F(\alpha)} p(\alpha)$. Roughly speaking, the second coordinate is as in the standard forcing for consistency of PFA, except with finite support and restricted to master conditions for nodes from the side conditions component. $\mathbb{A}$ is strongly proper for the set of transitive nodes, and so κ is preserved. Also, the forcing is proper for many models M^* such that $M^* \cap H(\kappa)$ is a countable node, which implies preservation of ω_1. Just as in the tree property application, cardinals in between are collapsed, and so κ becomes $\aleph_2$. Finally, PFA holds in the generic extension by $\mathbb{A}$ due to the Laver function iteration and the strong properness for the transitive nodes.

The main feature of this new proof is using finite support iteration instead of countable, and so not relying on preservation of properness under iterations (which fails for finite support). This difference is crucial when trying to generalize to obtain higher analogues of PFA. The standard proof uses preservation of properness under countable iterations. In order to meet ω_2-many dense sets, the natural way is to allow master conditions for both countable and ω_1-size models. But then preservation under iteration fails, making the problem seem hopeless for many years. Neeman's results avoid this obstacle, clearing the way to a new strategy for higher analogues. Of course, more work has to be done, since the original $\mathbb{P}_{\mathrm{side}}$ just preserves ω_1 and the supercompact cardinal, which becomes ω_2. And since PFA implies that the continuum is ω_2, a higher analogue will not be an actual strengthening of PFA.

In an upcoming paper, Neeman defines a modification of $\mathbb{P}_{side}$ that includes non elementary countable nodes in the side conditions. In this way three cardinals are preserved: ω_1, ω_2, and the supercompact, which will become ω_3. The author defines a higher analogue of properness: $\{\omega, \omega_1\}$-proper posets, by requiring the existence of master conditions for both countable and ω_1-size models. This is a fairly broad subclass of proper posets; for example it includes all c.c.c. posets and all two-sized side conditions. He then shows that starting from a supercompact cardinal, there is a forcing extension in which for every $\{\omega, \omega_1\}$-proper poset $\mathbb{Q}$ and for every collection of ω_2 many dense subsets, there is a filter for $\mathbb{Q}$ meeting these sets.

There are many remaining open problems about structures of size ω_2, that are the natural generalizations of classical results at ω_1. $\mathbb{P}_{\mathrm{side}}$ provides a promising strategy for attacking these problems. I expect there will be a lot more exciting developments in connection to this important type of forcing in extending the rich theory that exists at ω_1 to objects of size ω_2.

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